International Mathematics Competition for University Students
July 25–30 2009, Budapest, Hungary

Day 1

Problem 1.
Suppose that f and g are real-valued functions on the real line and $f(r) \le g(r)$ for every rational r. Does this imply that $f(x) \le g(x)$ for every real x if

a) f and g are non-decreasing?
b) f and g are continuous?

Problem 2.
Let A, B and C be real square matrices of the same size, and suppose that A is invertible. Prove that if $(A-B)C = BA^{-1}$, then $C(A-B) = A^{-1}B$.

Problem 3.
In a town every two residents who are not friends have a friend in common, and no one is a friend of everyone else. Let us number the residents from 1 to n and let a_i be the number of friends of the i-th resident. Suppose that $\sum_{i=1}^{n} a_i^2 = n^2 - n$. Let k be the smallest number of residents (at least three) who can be seated at a round table in such a way that any two neighbors are friends. Determine all possible values of k.

Problem 4.
Let $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ be a complex polynomial. Suppose that $1 = c_0 \ge c_1 \ge \cdots \ge c_n \ge 0$ is a sequence of real numbers which is convex (i.e. $2c_k \le c_{k-1} + c_{k+1}$ for every $k = 1, 2, \ldots, n-1$), and consider the polynomial

$$q(z) = c_0 a_0 + c_1 a_1 z + c_2 a_2 z^2 + \cdots + c_n a_n z^n.$$

Prove that

$$\max_{|z|\le 1} \big|q(z)\big| \le \max_{|z|\le 1} \big|p(z)\big|.$$

Problem 5.
Let n be a positive integer. An *n-simplex* in $\mathbb{R}^n$ is given by $n+1$ points $P_0, P_1, \ldots, P_n$, called its *vertices*, which do not all belong to the same hyperplane. For every n-simplex S we denote by $v(S)$ the volume of S, and we write $C(S)$ for the center of the unique sphere containing all the vertices of S.

Suppose that P is a point inside an n-simplex S. Let S_i be the n-simplex obtained from S by replacing its i-th vertex by P. Prove that

$$v(S_0)C(S_0) + v(S_1)C(S_1) + \cdots + v(S_n)C(S_n) = v(S)C(S).$$