International Mathematics Competition for University Students
July 25–30 2009, Budapest, Hungary

Day 1

Problem 1.
Suppose that f and g are real-valued functions on the real line and $f(r) \le g(r)$ for every rational r. Does this imply that $f(x) \le g(x)$ for every real x if

a) f and g are non-decreasing?

b) f and g are continuous?

Solution. a) No. Counter-example: f and g can be chosen as the characteristic functions of $[\sqrt{3}, \infty)$ and $(\sqrt{3}, \infty)$, respectively.

b) Yes. By the assumptions $g - f$ is continuous on the whole real line and nonnegative on the rationals. Since any real number can be obtained as a limit of rational numbers we get that $g - f$ is nonnegative on the whole real line.

Problem 2.
Let A, B and C be real square matrices of the same size, and suppose that A is invertible. Prove that if $(A-B)C = BA^{-1}$, then $C(A-B) = A^{-1}B$.

Solution. A straightforward calculation shows that $(A-B)C = BA^{-1}$ is equivalent to $AC - BC - BA^{-1} + AA^{-1} = I$, where I denotes the identity matrix. This is equivalent to $(A-B)(C + A^{-1}) = I$. Hence, $(A-B)^{-1} = C + A^{-1}$, meaning that $(C + A^{-1})(A - B) = I$ also holds. Expansion yields the desired result.

Problem 3.
In a town every two residents who are not friends have a friend in common, and no one is a friend of everyone else. Let us number the residents from 1 to n and let a_i be the number of friends of the i-th resident. Suppose that $\sum_{i=1}^{n} a_i^2 = n^2 - n$. Let k be the smallest number of residents (at least three) who can be seated at a round table in such a way that any two neighbors are friends. Determine all possible values of k.

Solution. Let us define the simple, undirected graph G so that the vertices of G are the town's residents and the edges of G are the friendships between the residents. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ denote the vertices of G; a_i is degree of v_i for every i. Let $E(G)$ denote the edges of G. In this terminology, the problem asks us to describe the length k of the shortest cycle in G.

Let us count the walks of length 2 in G, that is, the ordered triples (v_i, v_j, v_l) of vertices with $v_iv_j, v_jv_l \in E(G)$ ($i = l$ being allowed). For a given j the number is obviously a_j^2, therefore the total number is $\sum_{i=1}^{n} a_i^2 = n^2 - n$.

Now we show that there is an injection f from the set of ordered pairs of distinct vertices to the set of these walks. For $v_iv_j \notin E(G)$, let $f(v_i, v_j) = (v_i, v_l, v_j)$ with arbitrary l such that $v_iv_l, v_lv_j \in E(G)$. For $v_iv_j \in E(G)$, let $f(v_i, v_j) = (v_i, v_j, v_i)$. f is an injection since for $i \ne l$, (v_i, v_j, v_l) can only be the image of (v_i, v_l), and for $i = l$, it can only be the image of (v_i, v_j).

Since the number of ordered pairs of distinct vertices is $n^2 - n$, $\sum_{i=1}^{n} a_i^2 \ge n^2 - n$. Equality holds iff f is surjective, that is, iff there is exactly one l with $v_iv_l, v_lv_j \in E(G)$ for every i, j with $v_iv_j \notin E(G)$ and there is no such l for any i, j with $v_iv_j \in E(G)$. In other words, iff G contains neither C_3 nor C_4 (cycles of length 3 or 4), that is, G is either a forest (a cycle-free graph) or the length of its shortest cycle is at least 5.

It is easy to check that if every two vertices of a forest are connected by a path of length at most 2, then the forest is a star (one vertex is connected to all others by an edge). But G has n vertices, and none of them has degree $n-1$. Hence G is not forest, so it has cycles. On the other hand, if the length of a cycle C of G is at least 6 then it has two vertices such that both arcs of C connecting them are longer than 2. Hence there is a path connecting them that is shorter than both arcs. Replacing one of the arcs by this path, we have a closed walk shorter than C. Therefore length of the shortest cycle is 5.

Finally, we must note that there is at least one G with the prescribed properties – e.g. the cycle C_5 itself satisfies the conditions. Thus 5 is the sole possible value of k.

Problem 4.
Let $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ be a complex polynomial. Suppose that $1 = c_0 \geq c_1 \geq \cdots \geq c_n \geq 0$ is a sequence of real numbers which is convex (i.e. $2c_k \leq c_{k-1} + c_{k+1}$ for every $k = 1, 2, \dots, n-1$), and consider the polynomial

$$q(z) = c_0 a_0 + c_1 a_1 z + c_2 a_2 z^2 + \cdots + c_n a_n z^n.$$

Prove that

$$\max_{|z|\leq 1} \big|q(z)\big| \leq \max_{|z|\leq 1} \big|p(z)\big|.$$

Solution. The polynomials p and q are regular on the complex plane, so by the Maximum Principle, $\max_{|z|\leq 1} |q(z)| = \max_{|z|=1} |q(z)|$, and similarly for p. Let us denote $M_f = \max_{|z|=1} |f(z)|$ for any regular function f. Thus it suffices to prove that $M_q \leq M_p$.

First, note that we can assume $c_n = 0$. Indeed, for $c_n = 1$, we get $p = q$ and the statement is trivial; otherwise, $q(z) = c_n p(z) + (1 - c_n) r(z)$, where $r(z) = \sum_{j=0}^{n} \frac{c_j - c_n}{1 - c_n} a_j z^j$. The sequence $c'_j = \frac{c_j - c_n}{1 - c_n}$ also satisfies the prescribed conditions (it is a positive linear transform of the sequence c_n with $c'_0 = 1$), but $c'_n = 0$ too, so we get $M_r \leq M_p$. This is enough: $M_q = |q(z_0)| \leq c_n |p(z_0)| + (1 - c_n)|r(z_0)| \leq c_n M_p + (1 - c_n) M_r \leq M_p$.

Using the Cauchy formulas, we can express the coefficients a_j of p from its values taken over the positively oriented circle $S = \{|z| = 1\}$:

$$a_j = \frac{1}{2\pi i} \int_S \frac{p(z)}{z^{j+1}} \mathrm{d}z = \frac{1}{2\pi} \int_S \frac{p(z)}{z^j} |dz|$$

for $0 \leq j \leq n$, otherwise

$$\int_S \frac{p(z)}{z^j} |dz| = 0.$$

Let us use these identities to get a new formula for q, using only the values of p over S:

$$2\pi \cdot q(w) = \sum_{j=0}^{n} c_j \left(\int_S p(z) z^{-j} |dz| \right) w^j.$$

We can exchange the order of the summation and the integration (sufficient conditions to do this obviously apply):

$$2\pi \cdot q(w) = \int_S \left(\sum_{j=0}^{n} c_j (w/z)^j \right) p(z) |dz|.$$

It would be nice if the integration kernel (the sum between the brackets) was real. But this is easily arranged – for $-n \leq j \leq -1$, we can add the conjugate expressions, because by the above remarks, they are zero anyway:

$$2\pi \cdot q(w) = \sum_{j=0}^{n} c_j \left(\int_S p(z) z^{-j} |dz| \right) w^j = \sum_{j=-n}^{n} c_{|j|} \left(\int_S p(z) z^{-j} |dz| \right) w^j,$$

$$2\pi \cdot q(w) = \int_S \left(\sum_{j=-n}^{n} c_{|j|} (w/z)^j \right) p(z) |dz| = \int_S K(w/z) p(z) |dz|,$$

where

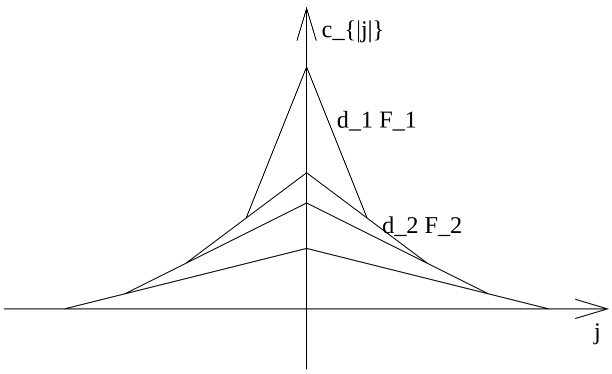
$$K(u) = \sum_{j=-n}^{n} c_{|j|} u^j = c_0 + 2 \sum_{j=1}^{n} c_j \Re(u^j)$$

for $u \in S$.

Let us examine $K(u)$. It is a real-valued function. Again from the Cauchy formulas, $\int_S K(u) |\mathrm{d}u| = 2\pi c_0 = 2\pi$. If $\int_S |K(u)| |\mathrm{d}u| = 2\pi$ still holds (taking the absolute value does not increase the integral), then for every w:

$$2\pi |q(w)| = \left| \int_S K(w/z) p(z) |dz| \right| \leq \int_S |K(w/z)| \cdot |p(z)| |dz| \leq M_p \int_S |K(u)| |du| = 2\pi M_p;$$

this would conclude the proof. So it suffices to prove that $\int_S |K(u)| |\mathrm{d}u| = \int_S K(u) |\mathrm{d}u|$, which is to say, K is non-negative.



Now let us decompose K into a sum using the given conditions for the numbers c_j (including $c_n = 0$). Let $d_k = c_{k-1} - 2c_k + c_{k+1}$ for $k = 1, \ldots, n$ (setting $c_{n+1} = 0$); we know that $d_k \geq 0$. Let $F_k(u) = \sum_{j=-k+1}^{k-1} (k - |j|) u^j$. Then $K(u) = \sum_{k=1}^{n} d_k F_k(u)$ by easy induction (or see Figure for a graphical illustration). So it suffices to prove that $F_k(u)$ is real and $F_k(u) \geq 0$ for $u \in S$. This is reasonably well-known (as $\frac{F_k}{k}$ is the Fejér kernel), and also very easy:

$$F_k(u) = (1 + u + u^2 + \cdots + u^{k-1})(1 + u^{-1} + u^{-2} + \cdots + u^{-(k-1)}) =$$
$$= (1 + u + u^2 + \cdots + u^{k-1})\overline{(1 + u + u^2 + \cdots + u^{k-1})} = |1 + u + u^2 + \cdots + u^{k-1}|^2 \geq 0$$

This completes the proof.

Problem 5.
Let n be a positive integer. An *n-simplex* in $\mathbb{R}^n$ is given by $n+1$ points $P_0, P_1, \ldots, P_n$, called its *vertices*, which do not all belong to the same hyperplane. For every n-simplex S we denote by $v(S)$ the volume of S, and we write $C(S)$ for the center of the unique sphere containing all the vertices of S.

Suppose that P is a point inside an n-simplex S. Let S_i be the n-simplex obtained from S by replacing its i-th vertex by P. Prove that

$$v(S_0)C(S_0) + v(S_1)C(S_1) + \cdots + v(S_n)C(S_n) = v(S)C(S).$$

Solution 1. We will prove this by induction on n, starting with $n = 1$. In that case we are given an interval $[a, b]$ with a point $p \in (a, b)$, and we have to verify

$$(b-p)\frac{b+p}{2} + (p-a)\frac{p+a}{2} = (b-a)\frac{b+a}{2},$$

which is true.

Now let assume the result is true for $n-1$ and prove it for n. We have to show that the point

$$X = \sum_{j=0}^{n} \frac{v(S_j)}{v(S)} O(S_j)$$

has the same distance to all the points $P_0, P_1, \ldots, P_n$. Let $i \in \{0, 1, 2, \ldots, n\}$ and define the sets $M_i = \{P_0, P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_n\}$. The set of all points having the same distance to all points in M_i is a line h_i orthogonal to the hyperplane E_i determined by the points in M_i. We are going to show that X lies on every h_i. To do so, fix some index i and notice that

$$X = \frac{v(S_i)}{v(S)} O(S_i) + \frac{v(S) - v(S_i)}{v(S)} \cdot \underbrace{\sum_{j \neq i} \frac{v(S_j)}{v(S) - v(S_i)} O(S_j)}_{Y}$$

and $O(S_i)$ lies on h_i, so that it is enough to show that Y lies on h_i.

A map $f : \mathbb{R}_{>0} \to \mathbb{R}^n$ will be called *affine* if there are points $A, B \in \mathbb{R}^n$ such that $f(\lambda) = \lambda A + (1-\lambda)B$. Consider the ray g starting in P_i and passing through P. For $\lambda > 0$ let $P_\lambda = (1-\lambda)P + \lambda P_i$, so that P_λ is an affine function describing the points of g. For every such λ let S_j^λ be the n-simplex obtained from S by replacing the j-th vertex by P_λ. The point $O(S_j^\lambda)$ is the intersection of the fixed line h_j with the hyperplane orthogonal to

g and passing through the midpoint of the segment $\overline{P_i P_\lambda}$ which is given by an affine function. This implies that also $O(S_j^\lambda)$ is an affine function. We write $\varphi_j = \frac{v(S_j)}{v(S)-s(S_i)}$, and then

$$Y_\lambda = \sum_{j \neq i} \varphi_j O(S_j^\lambda)$$

is an affine function. We want to show that $Y_\lambda \in h_i$ for all λ (then specializing to $\lambda = 1$ gives the desired result). It is enough to do this for two different values of λ.

Let g intersect the sphere containing the vertices of S in a point Z; then $Z = P_\lambda$ for a suitable $\lambda > 0$, and we have $O(S_j^\lambda) = O(S)$ for all j, so that $Y_\lambda = O(S) \in h_i$. Now let g intersect the hyperplane E_i in a point Q; then $Q = P_\lambda$ for some $\lambda > 0$, and Q is different from Z. Define T to be the $(n-1)$-simplex with vertex set M_i, and let T_j be the $(n-1)$-simplex obtained from T by replacing the vertex P_j by Q. If we write v' for the volume of $(n-1)$-simplices in the hyperplane E_i, then

$$\begin{aligned}\frac{v'(T_j)}{v'(T)} &= \frac{v(S_j^\lambda)}{v(S)} = \frac{v(S_j^\lambda)}{\sum_{k \neq i} v(S_k^\lambda)} \\ &= \frac{\lambda v(S_j)}{\sum_{k \neq i} \lambda v(S_k)} = \frac{v(S_j)}{v(S) - v(S_i)} = \varphi_j.\end{aligned}$$

If p denotes the orthogonal projection onto E_i then $p(O(S_j^\lambda)) = O(T_j)$, so that $p(Y_\lambda) = \sum_{j \neq i} \varphi_j O(T_j)$ equals $O(T)$ by induction hypothesis, which implies $Y_\lambda \in p^{-1}(O(T)) = h_i$, and we are done.

Solution 2. For $n = 1$, the statement is checked easily.

Assume $n \geq 2$. Denote $O(S_j) - O(S)$ by q_j and $P_j - P$ by p_j. For all distinct j and k in the range $0, \ldots, n$ the point $O(S_j)$ lies on a hyperplane orthogonal to p_k and P_j lies on a hyperplane orthogonal to q_k. So we have

$$\begin{cases} \langle p_i, q_j - q_k \rangle = 0 \\ \langle q_i, p_j - p_k \rangle = 0 \end{cases}$$

for all $j \neq i \neq k$. This means that the value $\langle p_i, q_j \rangle$ is independent of j as long as $j \neq i$, denote this value by λ_i. Similarly, $\langle q_i, p_j \rangle = \mu_i$ for some μ_i. Since $n \geq 2$, these equalities imply that all the λ_i and μ_i values are equal, in particular, $\langle p_i, q_j \rangle = \langle p_j, q_i \rangle$ for any i and j.

We claim that for such p_i and q_i, the volumes

$$V_j = |\det(p_0, \ldots, p_{j-1}, p_{j+1}, \ldots, p_n)|$$

and

$$W_j = |\det(q_0, \ldots, q_{j-1}, q_{j+1}, \ldots, q_n)|$$

are proportional. Indeed, first assume that $p_0, \ldots, p_{n-1}$ and $q_0, \ldots, q_{n-1}$ are bases of $\mathbb{R}^n$, then we have

$$\begin{aligned} V_j &= \frac{1}{|\det(q_0, \ldots, q_{n-1})|} \left| \det\left((\langle p_k, q_l \rangle) \right)_{\substack{k \neq j \\ l < n}} \right| = \\ &= \frac{1}{|\det(q_0, \ldots, q_{n-1})|} \left| \det((\langle p_k, q_l \rangle))_{\substack{l \neq j \\ k < n}} \right| = \left| \frac{\det(p_0, \ldots, p_{n-1})}{\det(q_0, \ldots, q_{n-1})} \right| W_j. \end{aligned}$$

If our assumption did not hold after any reindexing of the vectors p_i and q_i, then both p_i and q_i span a subspace of dimension at most $n-1$ and all the volumes are 0.

Finally, it is clear that $\sum q_j W_j / \det(q_0, \ldots, q_n) = 0$: the weight of p_j is the height of 0 over the hyperplane spanned by the rest of the vectors q_k relative to the height of p_j over the same hyperplane, so the sum is parallel to all the faces of the simplex spanned by $q_0, \ldots, q_n$. By the argument above, we can change the weights to the proportional set of weights $V_j / \det(p_0, \ldots, p_n)$ and the sum will still be 0. That is,

$$\begin{aligned} 0 &= \sum q_j \frac{V_j}{\det(p_0, \ldots, p_n)} = \sum (O(S_j) - O(S)) \frac{v(S_j)}{v(S)} = \\ &= \frac{1}{v(S)} \left(\sum O(S_j) v(S_j) - O(S) \sum v(S_j) \right) = \frac{1}{v(S)} \left(\sum O(S_j) v(S_j) - O(S) v(S) \right), \end{aligned}$$

q.e.d.